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Pressure-based dispatch
for shared autonomous
vehicles

Final Report

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| 16. Abstract (Limit: 250 words) Shared autonomous vehicle (SAV) technology is rapidly maturing, with two companies (Uber and Waymo) already testing SAV services in cities in the US. Due to the point-to-point service and lack of a driver, SAV service costs could be similar to that of personal vehicles, resulting in major mode choice changes for daily travel. A major issue for SAV operators is the SAV dispatch problem, i.e., how to optimally assign vehicles to waiting passengers. SAV dispatch is essentially a vehicle routing problem, which is NP-hard, and is complicated by fleets measured in thousands of vehicles in typical cities. Previous studies have attempted to quantify the number of passengers served per SAV using agent-based simulation studies on realistic networks, with a variety of results that highly depend on the heuristic chosen for SAV dispatch. Ideally, the optimal SAV dispatch strategy would serve as many passengers as any other policy. This project created a max-pressure dispatch policy, which was analytically proven by showing stability in the number of unserved passengers through a Lyapunov function. Essentially, the work analytically compared the serviceable demand from the max-pressure dispatch to the demand that could be served by any other dispatch policy. The max-pressure policy relied on a planning horizon; as the horizon grows to infinity, the policy becomes arbitrarily close to any sequence of SAV movements that can serve given demand rates. | | | |
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Pressure-based dispatch for shared autonomous vehicles

FINAL REPORT

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1 Executive Summary

The development of automated vehicle technology has led to prototype shared autonomous vehicle (SAV) transportation systems. Essentially driverless ridesharing, such systems could in the future offer low-cost mobility-on-demand to travelers that may be sufficiently low-cost and convenient so as to replace personal vehicle ownership. Consequently, the efficiency of SAV systems, particularly in terms of passenger service, could become highly important in the future. This report develops an analytical throughput-optimal dispatch policy for SAVs. In other words, this policy is guaranteed to serve at least as much demand as any other dispatch policy. This report proceeds by first developing a store-and-forward queueing model for the SAV system. Although the sparseness of typical demand patterns creates significant overhead in unused variables, the simplicity of the solution may compensate. The report develops a SAV dispatch based on a planning horizon of scheduling current and future SAV trips to serve waiting passengers. The analytical work proves that when the planning horizon is sufficiently large, the max-pressure policy can come arbitrarily close to serving any demand that can be served by any other policy.

2 Introduction

Due to the rapid development of automated vehicle technology, Waymo and Uber have started to trial shared autonomous vehicle (SAV) service in several cities in the United States. Due to the lack of a driver, SAVs could eventually provide point-to-point transportation at a cost similar to that of personal vehicles. Consequently, travelers may choose to rely on SAVs rather than personal vehicle ownership for their daily transportation needs. Previous studies (e.g. Fagnant and Kockelman, 2014; Fagnant et al., 2015; Spieser et al., 2014) have constructed simulations of SAV interactions with travelers on city networks and found that SAVs could replace between 3 to 11 personal vehicles. However, results are highly dependent on the passenger-to-vehicle matching used. Since vehicle routing problems are in general NP-hard, most studies have used heuristics to dispatch SAVs to passengers. Other challenges include the additional congestion caused by SAV route patterns and empty travel (Levin et al., 2017). The combined route choice and dispatch problem can be formulated as a linear program (for continuous SAV flows) (Levin, 2017) but requires future knowledge of travel demand.

The purpose of this report is to address the SAV dispatch problem through the notion of stability from max-pressure control of traffic signals (Varaiya, 2013). For traffic networks, stability occurs when the number of vehicles in the network remains bounded in expectation. Inefficient signal timings or sufficiently high demand prevent stability. For a system of SAVs, the number of vehicles in the network is constant (the fleet size), but the number of waiting travelers could grow arbitrarily large if the fleet is too small to serve them. Ideally, the dispatch strategy for SAVs would maintain stability for the largest set of demand possible.

The contributions of this report are as follows: We construct a store-and-forward queueing model that represents the SAV system. We analytically derive the region of demands that could be stabilized for any SAV system, which may be used for determining the relationship between fleet size and demand. We analytically develop a max-pressure dispatch policy for SAVs and prove it has maximum-stability. In other words, if an average demand can be served by some dispatch policy, then the max-pressure dispatch will serve it.

3 Network model

3.1 Network definition

Consider a traffic network $\mathcal{G} = (\mathcal{N}, \mathcal{A})$ with set of nodes \mathcal{N} and set of links \mathcal{A} . SAVs travel through this network, interacting with passengers. Let $\mathcal{Z} \subset \mathcal{A}$ be the set of zones, which are a subset of the links because SAVs enter and exit zones to pick-up and drop-off travelers then proceed to their next assignment. Let \mathcal{A}_o be the set of non-zone links. Each time step, SAVs can move forward through the network towards zones. Let Γ_i^+ and Γ_i^- be the forwards and backwards stars of i , respectively. Assume without loss of generality that each link has a travel time of one time step (longer links can be separated into segments). Let $x_j^{rs}(t)$ be the number of SAVs on link j traveling from $r \in \mathcal{Z}$ to $s \in \mathcal{Z}$ at time t . Let $y_{ij}^{rs}(t)$ be the number of SAVs going from r to s moving from link i to link j at time t . Then $x_j^{rs}(t)$ evolves via conservation:

$$x_j^{rs}(t+1) = x_j^{rs}(t) + \sum_{i \in \mathcal{A}} y_{ij}^{rs}(t) - \sum_{k \in \mathcal{A}} y_{jk}^{rs}(t) \quad (1)$$

The variable $y_{ij}^{rs}(t)$ determine the route choice, and is constrained by SAVs on the link:

$$\sum_{k \in \Gamma_j^+} y_{jk}^{rs}(t) \leq x_j^{rs}(t) \quad (2)$$

At zones, SAV interactions are slightly different. SAVs can change their origin and destination at zones. Let $p_r(t)$ be the number of SAVs parked at r at time t . Then

$$p_r(t+1) = p_r(t) + \sum_{i \in \mathcal{A}} \sum_{q \in \mathcal{Z}} y_{ir}^{qr}(t) - \sum_{j \in \mathcal{A}} \sum_{s \in \mathcal{Z}} y_{rj}^{rs}(t) \quad (3)$$

Exiting vehicles is constrained by the number of parked vehicles:

$$\sum_{j \in \mathcal{A}} \sum_{s \in \mathcal{Z}} y_{rj}^{rs}(t) \leq p_r(t) \quad (4)$$

Equation (4) includes an implicit constraint. SAV movements are limited by the fleet size $F = \sum_{r \in \mathcal{Z}} p_r(0)$.

Let Φ_i^s be the free flow shortest path travel time from i to s . For this paper we assume that travel times are constant. However, we retain the network structure so that future work can consider congestion effects from other vehicles or even from SAV interactions.

3.2 Demand

Passengers have an origin $r \in \mathcal{Z}$ and a destination $s \in \mathcal{Z}$. Let $d^{rs}(t)$ be the demand (number of new passengers) at time t wishing to travel from r to s . $d^{rs}(t)$ are random independent identically distributed variables with mean \bar{d}^{rs} . We assume that passengers wait at their origin until being picked up. Let $w^{rs}(t)$ be the number of passengers waiting at r for travel to s . We use $\mathbf{d}(t)$ and $\mathbf{w}(t)$ to denote the vectors of demand and waiting passengers, respectively. Passengers can only depart r after being picked up, so $w^{rs}(t)$ evolves as follows:

$$w^{rs}(t+1) = w^{rs}(t) + d^{rs}(t) - \min \left\{ \sum_{j \in \mathcal{A}} y_{rj}^{rs}(t), w^{rs}(t) \right\} \quad (5)$$

because $\sum_{j \in \mathcal{A}} y_{rj}^{rs}(t)$ is the number of SAVs departing r for s at time t . Some of these SAVs may travel from r to s empty for rebalancing in response to actual or predicted demand.

Equation (5) implicitly assumes that SAV passengers will wait until they are picked up by a vehicle. In reality, travelers are likely to give up if the waiting time becomes excessive. However, by modeling passengers as continuing to wait, we ensure that we maximize stability, i.e. serve as many passengers as possible. We leave for future work the notion of stability and the corresponding SAV dispatch behavior when passengers have a maximum waiting time.

3.3 Max-pressure policy

We now define a max-pressure policy π^* , where the pressure is provided by current numbers of waiting passengers. This policy is defined by a *planning horizon* T . At each time step, optimal vehicle movement over the interval $[t, t + T]$ is decided. Then, vehicle assignments at t are actually used. This process is essentially a rolling horizon optimization heuristic. However, it will become clear from the proof of stability that this rolling horizon is necessary for stability. Essentially, the SAV dispatcher must plan ahead to rebalance vehicles as needed to satisfy current demand. This policy is found by solving the following linear program to allocate the fleet of SAVs to trips. Let $f^{rs}(t)$ be the number of SAVs departing from r to s at time t .

$$\max \quad \frac{1}{T} \sum_{\tau=1}^T \sum_{(r,s) \in \mathcal{Z}^2} w^{rs}(t) f^{rs}(t + \tau) \quad (6a)$$

$$\text{s.t.} \quad \sum_{s \in \mathcal{Z}} f^{rs}(t + \tau) \leq p_r(t + \tau) \quad \forall r \in \mathcal{Z}, \forall \tau \in [0, T] \quad (6b)$$

$$p_r(t + \tau + 1) = p_r(t + \tau) + \sum_{q \in \mathcal{Z}} f^{qr}(t + \tau - \Phi_q^r) - \sum_{s \in \mathcal{Z}} f^{rs}(t + \tau) + \sum_{q \in \mathcal{Z}} \sum_{i \in \mathcal{A}} x_i^{qr}(t + \tau - \Phi_i^r) \quad \forall r \in \mathcal{Z}, \forall \tau \in [0, T] \quad (6c)$$

$$f^{rs}(t + \tau) \geq 0 \quad \forall (r, z) \in \mathcal{Z}^2, \forall \tau \in [0, T] \quad (6d)$$

Notice that $p_r(t)$ is the number of SAVs parked at time step t , which is part of the state of the network. $p_r(t)$ will evolve over time depending on SAV trips and arriving vehicles, and the prediction is encoded into constraint (6c). SAVs on link i destined for r will arrive after Φ_i^q time, which explains the $\sum_{q \in \mathcal{Z}} \sum_{i \in \mathcal{A}} x_i^{qr}(t + \tau - \Phi_i^r)$ term in constraint (6c). After solving linear program (6), assign $f^{rs}(t)$ vehicles to depart from r for s at time step t . The remainder of the solution is discarded. SAV assignments for time $t + 1$ (and future times) will be re-optimized at the next time step using the updated realization of waiting travelers.

The second component of the policy is the route choice in the network. Recall that we assume that travel time are constant. Therefore, for every link i and every origin-destination pair (r, s) , do the following: find a link j such that $\Phi_i^s > \Phi_j^s$ (which exists by definition of the shortest path). Set $y_{ij}^{rs}(t) = x_i^{rs}(t)$ to move all SAVs on i traveling from r to s to j . By always moving vehicles towards their destination, the travel time from r to s will always be equal to Φ_r^s . Notice that this includes the conversion of the $f^{rs}(t)$ variables to SAV departures in terms of $y_{ri}^{rs}(t)$ for some link $i \in \Gamma_r^+$. SAVs departing from r to s are routed onto a link i with $\Phi_i^s < \Phi_r^s$.

We also propose a variation on π^* which accounts for future expected demand in addition to the current numbers of waiting passengers by replacing the objective function (6a) with

$$\min \frac{1}{T} \sum_{\tau=1}^T \sum_{(r,s) \in \mathcal{Z}^2} (w^{rs}(t) + \tau \bar{d}^{rs}) f^{rs}(t + \tau) \quad (7)$$

We refer to this variation as $\pi_{\bar{\mathbf{d}}}^*$ because it requires knowledge of the average demand $\bar{\mathbf{d}}$.

4 Stability proof

We use Varaiya (2013)'s definition of stability, which is that the average expected number of waiting travelers remains bounded for every time T .

Definition 1. *The stochastic queueing model is stable if there exists some $K < \infty$*

$$\mathbb{E} \left[\frac{1}{T} \sum_{t=1}^T \sum_{(r,s) \in \mathcal{Z}^2} w^{rs}(t) \right] \leq K \quad (8)$$

Theorem 2 of Leonardi et al. (2001) defines strong stability differently, which is actually equivalent to Definition 1:

Lemma 1. *Suppose that there exists a value function $\nu(\mathbf{w}(t))$ satisfying $0 \leq \nu(\mathbf{w}(t)) < \infty$ for all $\mathbf{w}(t)$ and*

$$\mathbb{E} [\nu(\mathbf{w}(t+1)) - \nu(\mathbf{w}(t)) | \mathbf{w}(t)] \leq \kappa - \epsilon |\mathbf{w}(t)| \quad (9)$$

for all $\mathbf{w}(t)$ for some $\kappa < \infty$. Then the queueing system satisfies the Definition 1 form of stability.

Proof. Uses part of the proof of Theorem 2 of Varaiya (2013). From equation (9),

$$\sum_{t=1}^T \mathbb{E} [\nu(\mathbf{w}(t+1)) - \nu(\mathbf{w}(t)) | \mathbf{w}(t)] \leq \kappa T - \epsilon \sum_{t=1}^T |\mathbb{E} [\mathbf{w}(t)]| \quad (10)$$

or equivalently

$$\mathbb{E} [\nu(\mathbf{w}(T+1))] - \mathbb{E} [\nu(\mathbf{w}(1))] \leq \kappa T - \epsilon \sum_{t=1}^T \mathbb{E} [\mathbf{w}(t)] \quad (11)$$

which yields

$$\epsilon \frac{1}{T} \sum_{t=1}^T |\mathbb{E} [\mathbf{w}(t)]| \leq \kappa + \frac{1}{T} \mathbb{E} [\nu(\mathbf{w}(1))] - \frac{1}{T} \mathbb{E} [\nu(\mathbf{w}(T+1))] \leq \kappa + \frac{1}{T} \mathbb{E} [\nu(\mathbf{w}(1))] \quad (12)$$

which implies stability condition (8). \square

4.1 Stability region

For any given fleet size, it is easily possible to input an average demand rate that cannot be stabilized. If the demand is sufficiently high, no SAV dispatch policy will be able to serve all travelers. Therefore it is necessary to characterize the stability region. We then show that the max-pressure policy will stabilize any demand within the stability region.

Let \bar{y}_{ij}^{rs} be an average rate of SAV flow from i to j from r to s , and let $\bar{\mathbf{y}}$ be the average flow vector. The system is stable if there exists \bar{y}_{ij}^{rs} satisfying the following constraints. First, the sum of the average flows cannot exceed the fleet size.

$$\sum_{(r,s) \in \mathcal{Z}^2} \sum_{(i,j) \in \mathcal{A}^2} \bar{y}_{ij}^{rs} \leq F \quad (13)$$

Flow must be conserved on links, i.e. the number of SAVs entering a link is equal to the number of SAVs exiting.

$$\sum_{i \in \Gamma_j^-} \bar{y}_{ij}^{rs} = \sum_{j \in \Gamma_j^+} \bar{y}_{jk}^{rs} \quad \forall (r,s) \in \mathcal{Z}^2, \forall j \in \mathcal{A}_o \quad (14)$$

Conservation of flow on centroids is different from the conservation of flow on links because SAV destinations can change:

$$\sum_{q \in \mathcal{Z}} \sum_{i \in \Gamma_r^-} \bar{y}_{ir}^{qr} = \sum_{s \in \mathcal{Z}} \sum_{j \in \Gamma_r^+} \bar{y}_{jr}^{rs} \quad \forall q \in \mathcal{Z} \quad (15)$$

Let $\bar{\mathcal{Y}}$ be the set of average SAV assignments $\bar{\mathbf{y}}$ satisfying constraints (13)–(15). $\bar{\mathcal{Y}}$ is the feasible set of stationary SAV dispatch assignments.

The last constraint defining the stability region is to satisfy all demand:

$$\sum_{i \in \Gamma_r^+} \bar{y}_{ri}^{rs} \geq \bar{d}^{rs} \quad \forall (r, s) \in \mathcal{Z}^2 \quad (16)$$

Let $\mathcal{D} = \{\mathbf{d} : \exists \bar{\mathbf{y}} \in \bar{\mathcal{Y}} \text{ satisfying (16)}\}$. Let \mathcal{D}^0 be the interior of \mathcal{D} , i.e. the set of demands for which constraints (13) and (16) are strict inequalities.

To connect $\bar{\mathbf{y}}$ with actual SAV movements, it is necessary to show that there exists a sequence of SAV dispatch assignments that will result in stationary average flows of $\bar{\mathbf{y}}$. This sequence will also become useful in the proof of stability. The minimum length of the planning horizon T in π^* is based on how close the sequence needs to approximate $\bar{\mathbf{y}}$, which will be seen in Section 4.3.

Proposition 1. *There exists a sequence $(\mathbf{y}(t))$ such that*

$$\lim_{T \rightarrow \infty} \frac{1}{T} \sum_{t=0}^T \mathbf{y}(t) = \bar{\mathbf{y}} \quad (17)$$

for any $\mathbf{p}(0), \mathbf{x}(0)$ satisfying

$$\sum_{r \in \mathcal{Z}} p_r(0) + \sum_{(r,s) \in \mathcal{Z}^2} \sum_{i \in \mathcal{A}} x_i^{rs}(0) = F \quad (18)$$

Proof. For any $\epsilon > 0$ and for all $(i, j) \in \mathcal{A}^2, (r, s) \in \mathcal{Z}^2$, there exists rational numbers $\gamma_{ij}^{rs} \in \mathbb{Q}$ such that $|\bar{y}_{ij}^{rs} - \gamma_{ij}^{rs}| < \epsilon$ with γ satisfying constraints (14) and (15) because there is a rational number between any two different real numbers. Let K_1 be the least common multiple of the denominators of all γ_{ij}^{rs} , then assuming the correct starting positions of vehicles, there exists a $\mathbf{y}(t)$ such that

$$\sum_{t=0}^{K_1} \mathbf{y}(t) = \gamma K_1 \quad (19)$$

since γK_1 is integer and satisfies the conservation of flow at centroids and other nodes from constraints (14) and (15). For $T > K_1$, repeating the $\mathbf{y}(t)$ assignment will also yield γK_1 . Moving the vehicles to their starting locations requires at most $K_2 = \max_{(r,s) \in \mathcal{Z}^2} \{\Phi_r^s\}$ time, so there exists a $\mathbf{y}(t)$ that is at most ϵ away from $\bar{\mathbf{y}}$ after $K_1 + K_2$ time. \square

Given a fixed $T \in \mathbb{N}$, the $\mathbf{y}(t)$ that has the minimum difference between a $\bar{\mathbf{y}}$ that is sufficient for demand

$\bar{\mathbf{d}} \in \mathcal{D}^0$ can be found by solving the following quadratic program:

$$\min \sum_{(i,j) \in \mathcal{A}^2} \sum_{(r,s) \in \mathcal{Z}^2} \left(\bar{y}_{ij}^{rs} - \frac{1}{T} \sum_{t=0}^T y_{ij}^{rs}(t) \right)^2 \quad (20a)$$

$$\text{s.t.} \quad \sum_{j \in \mathcal{A}} y_{ij}^{rs}(t) \leq x_i^{rs}(t) \quad \forall (r,s) \in \mathcal{Z}^2, \forall t \in [0, T] \quad (20b)$$

$$x_j^{rs}(t+1) = x_j^{rs}(t) + \sum_{i \in \mathcal{A}} y_{ij}^{rs}(t) - \sum_{k \in \mathcal{A}} y_{jk}^{rs}(t) \quad \forall (r,s) \in \mathcal{Z}^2, \forall j \in \mathcal{A}_o, \forall t \in [0, T] \quad (20c)$$

$$p_r(t+1) = p_r(t) + \sum_{q \in \mathcal{Z}} \sum_{i \in \mathcal{A}} y_{ir}^{qr}(t) - \sum_{s \in \mathcal{Z}} \sum_{j \in \mathcal{A}} y_{rj}^{rs}(t) \quad \forall r \in \mathcal{Z}, \forall t \in [0, T] \quad (20d)$$

$$y_{ij}^{rs}(t) \geq 0 \quad \forall (r,s) \in \mathcal{Z}^2, \forall (i,j) \in \mathcal{A}^2, \forall t \in [0, T] \quad (20e)$$

(13)–(16)

with $\mathbf{p}(0)$ given. Notice that $\bar{\mathbf{y}}$ is also a decision variable in program (20) because there may be multiple $\bar{\mathbf{y}} \in \bar{\mathcal{Y}}$ that are sufficient for the demand $\bar{\mathbf{d}}$. Unfortunately, if T is a decision variable the optimization problem is no longer quadratic. Still, since T has a single dimension, the minimum value needed to obtain an ϵ level of precision can be found through a binary search.

Proposition 2. *If $\mathbf{d} \notin \mathcal{D}$, then the system cannot be stabilized by some $\bar{\mathbf{y}} \in \bar{\mathcal{Y}}$.*

Proof. For every $\bar{\mathbf{y}} \in \bar{\mathcal{Y}}$ there exists some $(r,s) \in \mathcal{Z}^2$ such that $\sum_{i \in \Gamma_r^+} \bar{y}_{ri}^{rs} - \bar{d}^{rs} \geq \eta$ for some $\eta > 0$. Then on average, $w^{rs}(t)$ will increase by η per time step, which does not satisfy Definition 1 for stability. \square

Proposition 2 shows that if $\mathbf{d} \notin \mathcal{D}$, then a larger fleet size is needed to serve all demand. Equivalently, if the max-pressure policy is stable for all demand in \mathcal{D} , then it will have maximum stability. The difference between \mathcal{D} and \mathcal{D}^0 is that constraints (13) and (16) can hold with equality in \mathcal{D} . If some average demand is on the boundary of \mathcal{D} , any dispatch policy will likely result in a null recurrent Markov chain. In other words, any realization of the stochastic demand significant above $\bar{\mathbf{d}}$ will take on average infinite time to be served. We therefore focus on the demand in \mathcal{D}^0 . Notice that the minimum fleet size to satisfy (13)–(16) can be determined using a linear program:

$$\min \quad F = \epsilon + \sum_{(r,s) \in \mathcal{Z}^2} \sum_{(i,j) \in \mathcal{A}^2} \bar{y}_{ij}^{rs} \quad (21a)$$

$$\text{s.t.} \quad \sum_{q \in \mathcal{Z}} \sum_{i \in \Gamma_r^-} \bar{y}_{ir}^{qr} = \sum_{s \in \mathcal{Z}} \sum_{j \in \Gamma_r^+} \bar{y}_{jr}^{rs} \quad \forall r \in \mathcal{Z} \quad (21b)$$

$$\sum_{i \in \Gamma_j^-} \bar{y}_{ij}^{rs} = \sum_{j \in \Gamma_j^+} \bar{y}_{jk}^{rs} \quad \forall (r,s) \in \mathcal{Z}^2, \forall j \in \mathcal{A}_o \quad (21c)$$

$$\sum_{i \in \Gamma_r^+} \bar{y}_{ri}^{rs} \geq \bar{d}^{rs} + \epsilon \quad \forall (r,s) \in \mathcal{Z}^2 \quad (21d)$$

$$\bar{y}_{ij}^{rs} \geq 0 \quad \forall (r,s) \in \mathcal{Z}^2, \forall (i,j) \in \mathcal{A}^2 \quad (21e)$$

where $\epsilon > 0$ is the buffer to ensure that constraints (13) and (16) are strict inequalities. The size of the buffer desired is exogenous, but it will become apparent in the proof of stability that a smaller buffer requires a larger planning horizon.

4.2 Stability Lemmas

To assist with the proof of stability for the max-pressure policy, we first state and prove two lemmas extending the conditions for stability according to Definition 1. Because SAV rebalancing requires time to take effect,

large realizations of stochastic demand will result in a correspondingly large number of waiting passengers that may not be reduced for some time.

Lemma 2. *Suppose that there exists a $T \in \mathbb{N}$ and a $\kappa_1, \kappa_2 < \infty$ such that*

$$\mathbb{E} [\nu(\mathbf{w}(t+T)) - \nu(\mathbf{w}(t+T+1)) + \nu(\mathbf{w}(t+1)) - \nu(\mathbf{w}(t)) | \mathbf{w}(t)] \leq \kappa_1 \quad (22)$$

and

$$\mathbb{E} [\nu(\mathbf{w}(t+T+1)) - \nu(\mathbf{w}(t+T)) | \mathbf{w}(t)] \leq \kappa_2 - \epsilon |\mathbf{w}(t)| \quad (23)$$

then the system satisfies Definition 1 of stability.

Proof. From inequality (23),

$$\mathbb{E} [\nu(\mathbf{w}(t+T+1)) - \nu(\mathbf{w}(t+T)) | \mathbf{w}(t)] \leq \kappa_2 - \epsilon |\mathbf{w}(t)| \quad (24)$$

so

$$\begin{aligned} \mathbb{E} [\nu(\mathbf{w}(t+T+1)) - \nu(\mathbf{w}(t+1)) + \nu(\mathbf{w}(t)) - \nu(\mathbf{w}(t+T)) | \mathbf{w}(t)] \\ + \mathbb{E} [\nu(\mathbf{w}(t+1)) - \nu(\mathbf{w}(t)) | \mathbf{w}(t)] \leq \kappa_2 - \epsilon |\mathbf{w}(t)| \end{aligned} \quad (25)$$

Combining inequalities (22) and (25) yields

$$\mathbb{E} [\nu(\mathbf{w}(t+1)) - \nu(\mathbf{w}(t)) | \mathbf{w}(t)] \leq \kappa_1 + \kappa_2 - \epsilon |\mathbf{w}(t)| \quad (26)$$

which implies stability by Lemma 1. \square

Lemma 3. *Suppose that inequality (22) holds for all $\tau \in \mathbb{N}$, and there exists a $T \in \mathbb{N}$ and a function $\nu(\mathbf{w}(t))$ such that*

$$\mathbb{E} \left[\frac{1}{T} \sum_{\tau=1}^T (\nu(\mathbf{w}(t+\tau+1)) - \nu(\mathbf{w}(t+\tau))) | \mathbf{w}(t) \right] \leq \kappa_2 - \epsilon |\mathbf{w}(t)| \quad (27)$$

then the system satisfies Definition 1 of stability.

Proof. Since equation (22) holds for all $\tau \in \mathbb{N}$,

$$\mathbb{E} \left[\frac{1}{T} \sum_{\tau=1}^T (\nu(\mathbf{w}(t+\tau)) - \nu(\mathbf{w}(t+\tau+1)) + \nu(\mathbf{w}(t+1)) - \nu(\mathbf{w}(t))) | \mathbf{w}(t) \right] \leq \kappa_1 \quad (28)$$

but

$$\mathbb{E} \left[\frac{1}{T} \sum_{\tau=1}^T (\nu(\mathbf{w}(t+1)) - \nu(\mathbf{w}(t))) | \mathbf{w}(t) \right] = \mathbb{E} [(\nu(\mathbf{w}(t+1)) - \nu(\mathbf{w}(t))) | \mathbf{w}(t)] \quad (29)$$

so equation (28) can be rewritten as

$$\mathbb{E} \left[\frac{1}{T} \sum_{\tau=1}^T (\nu(\mathbf{w}(t+\tau)) - \nu(\mathbf{w}(t+\tau+1)) + \nu(\mathbf{w}(t+1)) - \nu(\mathbf{w}(t))) | \mathbf{w}(t) \right] \leq \kappa_1 \quad (30)$$

Let $\nu'(\mathbf{w}(t+T)) = \frac{1}{T} \sum_{\tau=1}^T \nu(\mathbf{w}(t+T))$. By assumption of inequality (27), $\nu'(\mathbf{w}(t+T))$ satisfies inequality (23). By inequality (30), $\nu'(\mathbf{w}(t+T))$ satisfies (22). By Lemma 2, the system is stable. \square

4.3 Stability of π^*

Lemma 4. *The value function $\nu(\mathbf{w}(t)) = \sum_{(r,s) \in \mathcal{Z}^2} (w^{rs}(t))^2$ with $\mathbf{y}(t) = \bar{\mathbf{y}}$ satisfies inequality (22) for all $T \in \mathbb{N}$.*

Proof. We will separate the terms in $\mathbb{E}[\nu(\mathbf{w}(t+T)) - \nu(\mathbf{w}(t+T+1)) + \nu(\mathbf{w}(t+1)) - \nu(\mathbf{w}(t)) | \mathbf{w}(t)]$ into $\mathbb{E}[\nu(\mathbf{w}(t+T)) - \nu(\mathbf{w}(t+T+1)) | \mathbf{w}(t)]$ and $\mathbb{E}[\nu(\mathbf{w}(t+1)) - \nu(\mathbf{w}(t)) | \mathbf{w}(t)]$. Observe that

$$\mathbb{E}[\mathbf{w}(t+1) - \mathbf{w}(t) | \mathbf{w}(t)] = \sum_{(r,s) \in \mathcal{Z}^2} \bar{d}^{rs} + \min \left\{ \sum_{i \in \mathcal{A}} \bar{y}_{ri}^{rs}, w^{rs}(t+1) \right\} - \min \left\{ \sum_{i \in \mathcal{A}} \bar{y}_{ri}^{rs}, w^{rs}(t) \right\} \quad (31)$$

$$= \bar{d}^{rs} \quad (32)$$

when $\mathbf{w}(t)$ is larger than $\sum_{i \in \mathcal{A}} \bar{y}_{ri}^{rs}$. Similarly

$$\begin{aligned} & \mathbb{E}[\mathbf{w}(t+T) - \mathbf{w}(t+T-1) | \mathbf{w}(t)] \\ &= - \sum_{(r,s) \in \mathcal{Z}^2} \bar{d}^{rs} + \min \left\{ \sum_{i \in \mathcal{A}} \bar{y}_{ri}^{rs}, w^{rs}(t+T) \right\} - \min \left\{ \sum_{i \in \mathcal{A}} \bar{y}_{ri}^{rs}, w^{rs}(t+T-1) \right\} \end{aligned} \quad (33)$$

$$= -\bar{d}^{rs} \quad (34)$$

Let $\boldsymbol{\delta}_1 = \mathbf{w}(t+1) - \mathbf{w}(t)$ and $\boldsymbol{\delta}_2 = \mathbf{w}(t+T) - \mathbf{w}(t+T+1)$. Then

$$|\mathbf{w}(t+1)|^2 - |\mathbf{w}(t)|^2 = (\boldsymbol{\delta}_1 + \mathbf{w}(t))^2 - (\mathbf{w}(t))^2 = (\boldsymbol{\delta}_1)^+ 2\boldsymbol{\delta}_1 \mathbf{w}(t) \quad (35)$$

and

$$|\mathbf{w}(t+T)|^2 - |\mathbf{w}(t+T+1)|^2 = (\mathbf{w}(t))^2 - (\boldsymbol{\delta}_2 + \mathbf{w}(t+T))^2 = (\boldsymbol{\delta}_2)^2 + 2\boldsymbol{\delta}_2 \mathbf{w}(t+T) \quad (36)$$

By equation (32), $\boldsymbol{\delta}_1 = \bar{\mathbf{d}}$ so

$$(\boldsymbol{\delta}_1)^2 + 2\boldsymbol{\delta}_1 \mathbf{w}(t) = |\bar{\mathbf{d}}|^2 + 2\bar{\mathbf{d}} \cdot \mathbf{w}(t) \quad (37)$$

By equation (34), $\boldsymbol{\delta}_2 = -\bar{\mathbf{d}}$ so

$$\mathbb{E} \left[(\boldsymbol{\delta}_2)^2 + 2\boldsymbol{\delta}_2 \mathbf{w}(t+T) | \mathbf{w}(t) \right] = \mathbb{E} \left[|\bar{\mathbf{d}}|^2 + 2\bar{\mathbf{d}} \cdot \mathbf{w}(t+T) | \mathbf{w}(t) \right] = -|\bar{\mathbf{d}}|^2 - 2\bar{\mathbf{d}} \cdot (\mathbf{w}(t) + T\bar{\mathbf{d}}) \quad (38)$$

Combining equations (37) and (38) yields a constant term of $-2\bar{\mathbf{d}} \cdot (T\bar{\mathbf{d}})$. \square

Proposition 3. *There exists a $M < \infty$ such that if $T > M$ then the max-pressure control using the planning horizon $[t, t+T]$ stabilizes the network for any $\bar{\mathbf{d}} \in \mathcal{D}^0$.*

Proof. The value function $\nu(\mathbf{w}(t)) = \sum_{(r,s) \in \mathcal{Z}^2} (w^{rs}(t))^2$, which satisfies equation (22) for SAV dispatch of $\bar{\mathbf{y}}$ by Lemma 4. We will show that the π^* policy performs better than $\bar{\mathbf{y}}$, then appeal to Lemma 3. We need to show that

$$\mathbb{E} \left[\frac{1}{T} \sum_{\tau=1}^T \sum_{(r,s) \in \mathcal{Z}^2} (w^{rs}(t+\tau+1))^2 - (w^{rs}(t+\tau))^2 | \mathbf{w}(t) \right] \leq \kappa - \epsilon |\mathbf{w}(t)| \quad (39)$$

for some $T < \infty$ and some $\epsilon > 0$. Let

$$\delta^{rs}(t) = w^{rs}(t+1) - w^{rs}(t) = d^{rs}(t) - \min \left\{ w^{rs}(t), \sum_{i \in \mathcal{A}} y_{ri}^{rs}(t) \right\} \quad (40)$$

Making the substitution $w^{rs}(t+1) = \delta^{rs}(t) - w^{rs}(t)$,

$$\begin{aligned} & \mathbb{E} \left[\frac{1}{T} \sum_{\tau=1}^T \sum_{(r,s) \in \mathcal{Z}^2} (w^{rs}(t+\tau+1))^2 - (w^{rs}(t+\tau))^2 \mid \mathbf{w}(t) \right] \\ &= \mathbb{E} \left[\frac{1}{T} \sum_{\tau=1}^T \sum_{(r,s) \in \mathcal{Z}^2} (\delta^{rs}(t+\tau))^2 + 2w^{rs}(t+\tau)\delta^{rs}(t+\tau) \mid \mathbf{w}(t) \right] \end{aligned} \quad (41)$$

Because $\mathbb{E}[\delta^{rs}(t) \mid \mathbf{w}(t)] \leq \bar{d}^{rs}$,

$$\mathbb{E} \left[\frac{1}{T} \sum_{\tau=1}^T \sum_{(r,s) \in \mathcal{Z}^2} (\delta^{rs}(t+\tau))^2 \mid \mathbf{w}(t) \right] \leq \sum_{(r,s) \in \mathcal{Z}^2} (\bar{d}^{rs})^2$$

which leaves $\mathbb{E} \left[\frac{1}{T} \sum_{\tau=1}^T \sum_{(r,s) \in \mathcal{Z}^2} 2w^{rs}(t+\tau)\delta^{rs}(t+\tau) \mid \mathbf{w}(t) \right]$ from equation (41).

$$\begin{aligned} & \mathbb{E} \left[\frac{1}{T} \sum_{\tau=1}^T \sum_{(r,s) \in \mathcal{Z}^2} 2w^{rs}(t+\tau)\delta^{rs}(t+\tau) \mid \mathbf{w}(t) \right] \\ &= \mathbb{E} \left[\frac{1}{T} \sum_{\tau=1}^T \sum_{(r,s) \in \mathcal{Z}^2} 2w^{rs}(t+\tau) \left(\bar{d}^{rs} - \min \left\{ w^{rs}(t+\tau), \sum_{i \in \mathcal{A}} y_{ri}^{rs}(t+\tau) \right\} \right) \mid \mathbf{w}(t) \right] \end{aligned} \quad (42)$$

$$\begin{aligned} &= \mathbb{E} \left[\frac{1}{T} \sum_{\tau=1}^T \sum_{(r,s) \in \mathcal{Z}^2} 2w^{rs}(t+\tau) \left(\bar{d}^{rs} - \sum_{i \in \mathcal{A}} y_{ri}^{rs}(t+\tau) \right) \mid \mathbf{w}(t) \right] \\ &+ \mathbb{E} \left[\frac{1}{T} \sum_{\tau=1}^T \sum_{(r,s) \in \mathcal{Z}^2} 2w^{rs}(t+\tau) \left(\sum_{i \in \mathcal{A}} y_{ri}^{rs}(t+\tau) - \min \left\{ w^{rs}(t+\tau), \sum_{i \in \mathcal{A}} y_{ri}^{rs}(t+\tau) \right\} \right) \mid \mathbf{w}(t) \right] \end{aligned} \quad (43)$$

and

$$\sum_{(r,s) \in \mathcal{Z}^2} w^{rs}(t+\tau) \left(\sum_{i \in \mathcal{A}} y_{ri}^{rs}(t+\tau) - \min \left\{ w^{rs}(t+\tau), \sum_{i \in \mathcal{A}} y_{ri}^{rs}(t+\tau) \right\} \right) \leq F^2 \quad (44)$$

because if $w^{rs}(t+\tau) \geq y_{ri}^{rs}(t+\tau)$ then $\sum_{i \in \mathcal{A}} y_{ri}^{rs}(t+\tau) - \min \left\{ w^{rs}(t+\tau), \sum_{i \in \mathcal{A}} y_{ri}^{rs}(t+\tau) \right\} = 0$.

$$\begin{aligned} & \mathbb{E} \left[\frac{1}{T} \sum_{\tau=1}^T \sum_{(r,s) \in \mathcal{Z}^2} w^{rs}(t+\tau) \left(\bar{d}^{rs} - \sum_{i \in \mathcal{A}} y_{ri}^{rs}(t+\tau) \right) \mid \mathbf{w}(t) \right] \\ &= \frac{1}{T} \sum_{\tau=1}^T \sum_{(r,s) \in \mathcal{Z}^2} (w^{rs}(t) + \tau \bar{d}^{rs}) \left(\bar{d}^{rs} - \sum_{i \in \mathcal{A}} y_{ri}^{rs}(t+\tau) \right) \\ &- \frac{1}{T} \sum_{\tau=1}^T \sum_{(r,s) \in \mathcal{Z}^2} \left(\sum_{\tau'=1}^T \min \left\{ w^{rs}(t+\tau), \sum_{i \in \mathcal{A}} y_{ri}^{rs}(t+\tau) \right\} \right) \left(\bar{d}^{rs} - \sum_{i \in \mathcal{A}} y_{ri}^{rs}(t+\tau) \right) \end{aligned} \quad (45)$$

and

$$\frac{1}{T} \sum_{\tau=1}^T \sum_{(r,s) \in \mathcal{Z}^2} \left(\sum_{\tau'=1}^T \min \left\{ w^{rs}(t+\tau), \sum_{i \in \mathcal{A}} y_{ri}^{rs}(t+\tau) \right\} \right) \left(\bar{d}^{rs} - \sum_{i \in \mathcal{A}} y_{ri}^{rs}(t+\tau) \right) \leq FT \left(\sum_{(r,s) \in \mathcal{Z}^2} \bar{d}^{rs} \right) \quad (46)$$

We can further simplify by observing that

$$\begin{aligned} & \frac{1}{T} \sum_{\tau=1}^T \sum_{(r,s) \in \mathcal{Z}^2} (w^{rs}(t) + \tau \bar{d}^{rs}) \left(\bar{d}^{rs} - \sum_{i \in \mathcal{A}} y_{ri}^{rs}(t + \tau) \right) \\ &= \frac{1}{T} \sum_{\tau=1}^T \sum_{(r,s) \in \mathcal{Z}^2} w^{rs}(t) \left(\bar{d}^{rs} - \sum_{i \in \mathcal{A}} \hat{y}_{ri}^{rs}(t + \tau) \right) + \frac{1}{T} \sum_{\tau=1}^T \sum_{(r,s) \in \mathcal{Z}^2} \tau \bar{d}^{rs} \left(\bar{d}^{rs} - \sum_{i \in \mathcal{A}} \hat{y}_{ri}^{rs}(t + \tau) \right) \end{aligned} \quad (47)$$

The second term of (47) can be bounded:

$$\frac{1}{T} \sum_{\tau=1}^T \sum_{(r,s) \in \mathcal{Z}^2} \tau \bar{d}^{rs} \left(\bar{d}^{rs} - \sum_{i \in \mathcal{A}} \hat{y}_{ri}^{rs}(t + \tau) \right) \leq T \sum_{(r,s) \in \mathcal{Z}^2} (\bar{d}^{rs})^2 \quad (48)$$

leaving $\frac{1}{T} \sum_{\tau=1}^T \sum_{(r,s) \in \mathcal{Z}^2} w^{rs}(t) \left(\bar{d}^{rs} - \sum_{i \in \mathcal{A}} y_{ri}^{rs}(t + \tau) \right)$.

By Proposition 1 there exists a sequence $\hat{\mathbf{y}}(t + \tau)$ such that $\lim_{T \rightarrow \infty} \frac{1}{T} \sum_{\tau=1}^T \hat{\mathbf{y}}(t + \tau) = \bar{\mathbf{y}}$. When π^* is used,

$$\frac{1}{T} \sum_{\tau=1}^T \sum_{(r,s) \in \mathcal{Z}^2} w^{rs}(t) \left(\bar{d}^{rs} - \sum_{i \in \mathcal{A}} y_{ri}^{rs*}(t + \tau) \right) \leq \frac{1}{T} \sum_{\tau=1}^T \sum_{(r,s) \in \mathcal{Z}^2} w^{rs}(t) \left(\bar{d}^{rs} - \sum_{i \in \mathcal{A}} \hat{y}_{ri}^{rs}(t + \tau) \right) \quad (49)$$

because $\hat{\mathbf{y}}$ is a feasible solution to problem (6). Since $\hat{\mathbf{y}}(t + \tau)$ is a sequence with limit $\bar{\mathbf{y}}$, for every $\eta > 0$ there exists an $M < \infty$ such that for all $T > M$, $\frac{1}{T} \sum_{\tau=1}^T \hat{\mathbf{y}}(t + \tau) \leq |\bar{\mathbf{y}} - \eta \mathbf{1}|$ (where $\mathbf{1}$ is the vector of 1's). Therefore

$$\frac{1}{T} \sum_{\tau=1}^T \sum_{(r,s) \in \mathcal{Z}^2} w^{rs}(t) \left(\bar{d}^{rs} - \sum_{i \in \mathcal{A}} \hat{y}_{ri}^{rs}(t + \tau) \right) \leq \sum_{(r,s) \in \mathcal{Z}^2} w^{rs}(t) \left(\bar{d}^{rs} - \sum_{i \in \mathcal{A}} \bar{y}_{ri}^{rs} + \eta \right) \quad (50)$$

Because $\bar{\mathbf{d}} \in \mathcal{D}^0$, there exists an $\epsilon > 0$ such that for all $T > M$, $\bar{d}^{rs} - \sum_{i \in \mathcal{A}} \bar{y}_{ri}^{rs} \leq -\epsilon$. Choose M large enough that $\eta - \epsilon \leq -\epsilon_2$ for some $\epsilon_2 > 0$. Then

$$\sum_{(r,s) \in \mathcal{Z}^2} w^{rs}(t) \left(\bar{d}^{rs} - \sum_{i \in \mathcal{A}} \bar{y}_{ri}^{rs} + \eta \right) \leq \sum_{(r,s) \in \mathcal{Z}^2} w^{rs}(t) (-\epsilon_2) \leq -\epsilon_2 |\mathbf{w}(t)| \quad (51)$$

which satisfies the conditions of Lemma 3 for stability. \square

The stability of $\pi_{\bar{\mathbf{d}}}^*$ can be obtained by substituting $\hat{\mathbf{y}}$ for \mathbf{y}^* into equation (47) before the expected demand is removed from the inequality by bounding.

5 Conclusions

This report developed an analytical max-pressure dispatch policy for SAVs, which was proven to have maximum stability. In other words, this dispatch policy creates as much passenger throughput as any other dispatch strategy for SAVs, including strategies that have yet to be developed. The key to this result was comparing the max-pressure dispatch to every other dispatch policy through the notion of the stable region, which is the set of average demand rates that can be served by any policy. The max-pressure dispatch developed here was proven to serve any demand in the stable region. In the future, the project team will work on numerical comparisons of the max-pressure dispatch policy to supplement the analytical results from this project.

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